# Ising model on a hyperbolic plane with a boundary

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## Abstract

A hyperbolic plane can be modeled by a structure called the enhanced binary tree. We study the ferromagnetic Ising model on top of the enhanced binary tree using the renormalization-group analysis in combination with transfer-matrix calculations. We find a reasonable agreement with Monte Carlo calculations on the transition point, and the resulting critical exponents suggest the mean-field surface critical behavior.

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Critical phenomena on a plane has been one of the most well-studied area in statistical physics. One obvious reason is that it is a plane that we can most easily visualize, but it is also because a two-dimensional (2D) system is often analytically tractable. It is true that one could say the same about one-dimensional (1D) systems as well, but collective orders in a low-dimensional system tend to be fragile against thermal disorders; it is from the dimensionality $d = 2$ for many model systems to exhibit a phase transition at a nonzero temperature $T > 0$. These remarks are best illustrated by the Ising model defined by the following Hamiltonian

$$H = -J \sum_{\langle ij \rangle} S_i S_j - h \sum_i S_i, \tag{1}$$

where $J > 0$ is the ferromagnetic interaction strength, and $h$ is the magnetic-field strength. The first summation is over every pair of nearest neighbors, and spin at site $k$ can take its value $S_k$ from $\pm 1$. As is well known, the 1D Ising model does not have any magnetic order at finite temperatures, while the 2D counterpart undergoes a continuous order-disorder transition at a finite coupling strength $K \equiv \beta J$. Here the inverse temperature is denoted as $\beta \equiv (k_B T)^{-1}$, where $k_B$ is the Boltzmann constant [1]. In this regard, the lower critical dimension of the Ising model is 2.

By planes, however, we do not always have to mean the flat geometry. There are many kinds of curved planes observable in physical or biological structures, and critical phenomena on such planes may exhibit intriguing features. One can assign the Gaussian curvature to a plane, which is positive (negative) when the plane looks like a part of a sphere (saddle). With a constant positive Gaussian curvature, the plane will be eventually closed to form a sphere, whose radius is inversely proportional to the size of the curvature. This implies that the curvature should vanish at every local point if we are to work with a very large system size. Even though there can remain global topological constraints determined by the positive curvature, many statistical-physical properties will converge to those of the flat geometry in the large-size limit. In the hyperbolic geometry with a negative constant curvature, on the other hand, the magnitude of the curvature is not necessarily coupled to the system size, and therefore this case is often regarded as more suitable to study effects of the curvature. The price is that the surface area $\mathcal{A}$ expands exponentially as its radial length scale grows. It immediately leads the boundary of the surface $\partial\mathcal{A}$ to expand at the same rate, so we find that the boundary fraction $\partial\mathcal{A}/\mathcal{A}$ never vanishes even in the large-size

limit. The thermodynamic limit is not uniquely defined for this reason. For example, one may use the periodic boundary condition [2] and the behavior will not necessarily be the same as with the open boundary condition. By neglecting the presence of the boundary, it has been argued by many authors that the Ising model will undergo a mean-field-like phase transition on a negatively curved plane: a Monte Carlo analysis away from the boundary suggested convergence to the mean-field exponents in Ref. [3], which was later supported by the corner transfer-matrix renormalization-group (RG) method [4]. The Ginzburg-Landau theory provides a qualitative explanation for this mean-field behavior when only the bulk part is considered [5]. For rigorous results under transitivity, one may refer to Ref. [6] and references therein.

By accepting this nonvanishing boundary as a part of physics, it becomes possible to study nontrivial statistical-physical properties due to the boundary, as has been done in Refs. [5, 7–9]. A commonly used approach to simulate such a plane is to begin with hyperbolic tessellation with regular polygons and truncate the lattice generation at a certain layer [3, 10]. An advantage is that it makes every point equivalent except at the boundary layer. But there is also an alternative lattice structure for such a plane, called the enhanced binary tree (EBT) as shown in Fig. 1(a) [11]. It is obtained by adding links between branches in a binary tree [Fig. 1(b)], so the EBT itself is not a tree, strictly speaking. Although it is not a uniform tiling of a hyperbolic plane but made up of triangles and tetragons, this structure is actually easier to study analytically: for the bond-percolation problem, for example, it is possible to argue that the emergence of a single giant cluster occurs at a critical occupation probability $p = p_c = 1/2$ [12, 13]. This analytic tractability is particularly important because numerical calculations suffer from the exponential growth of the system size and therefore can only give very rough estimates. There can be in fact one more transition point where the correlation diverges in percolation or the Ising model [5, 9], but in this work, we focus on the order-disorder transition where the order parameter becomes nonzero. The simple binary tree in Fig. 1(b), for example, cannot have the latter type of order-disorder transition at any finite temperatures [5].

This Brief Report is intended to extend the analytic approaches for percolation to the Ising model on the EBT. We mainly rely on real-space RG methods and compare the results with the numerical data obtained by Monte Carlo (MC) calculations. Combined with the transfer-matrix method, this approximate RG calculation allows us to estimate the critical

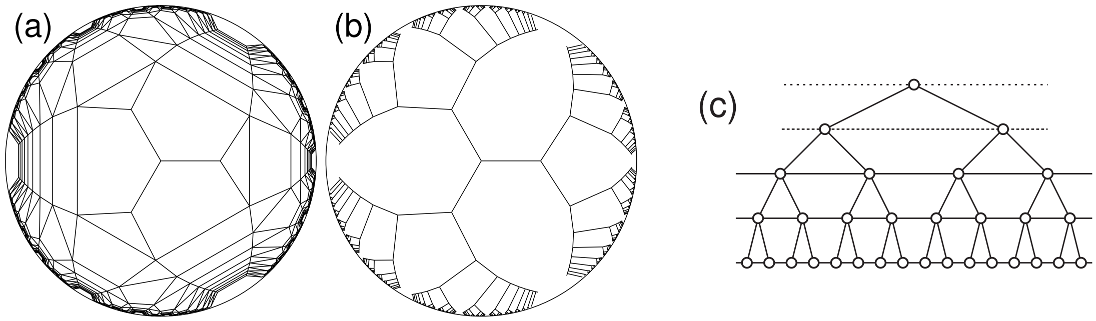


FIG. 1: Schematic descriptions of (a) the EBT and (b) the simple binary tree, drawn on the Poincaré disks. The number of layers is $L = 10$ in both the cases. (c) The actual structure used in our MC calculations. Here $L = 4$ is shown as an example, and the structure is extended further down for larger $L$. In (c), we remove the two upper most horizontal connections since they either make a self-link (the top dotted line) or double links (the dashed lines) under the periodic boundary condition in the horizontal direction.

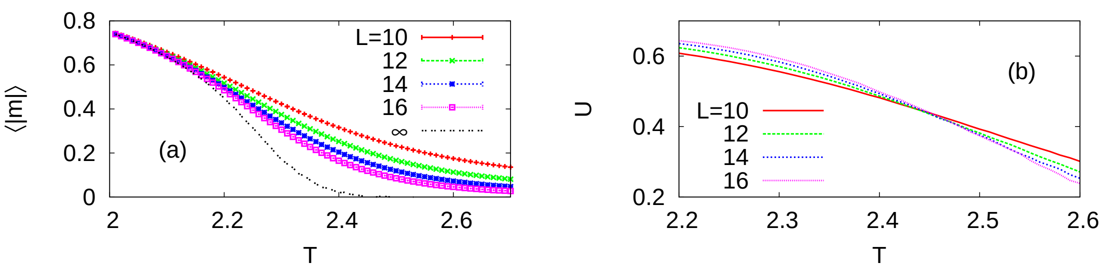


FIG. 2: (Color online) (a) Magnetic order parameter and (b) Binder's cumulant of the Ising model on the EBT structures with different sizes. The symbol $\infty$ in panel (a) means extrapolated values according to Eq. (2).

point and critical indices as well. The results suggest the mean-field critical behavior of free surfaces.

Let us begin with numerical methods used in this work, and give a very rough estimate of the critical coupling $K_c$ of the EBT. For MC calculations, an EBT structure is constructed with a certain number of layers, $L$, as shown in Fig. 1(c), where a periodic boundary condition is imposed in the horizontal direction. The system size is then given as $N = 2^{L+1} - 1$, which is an exponential function of $L$ as mentioned above. The Ising Hamiltonian [Eq. (1)] is simulated by the Wolff single-cluster algorithm [14]. The magnetic order parameter is

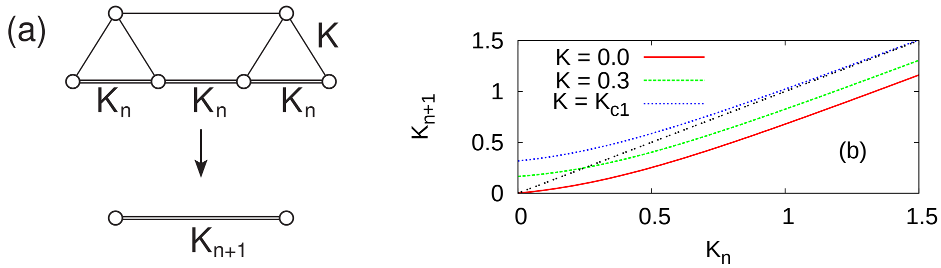


FIG. 3: (Color online) (a) Recursion scheme based on the finite-lattice method. The double lines represent coarse-grained effective couplings. (b) Iteration by Eq. (5) at various $K$'s. The straight line has slope 1, i.e., $K_{n+1} = K_n$.

defined by

$$\langle |m| \rangle \equiv \left\langle \left| \frac{1}{N} \sum_i^N S_i \right| \right\rangle,$$

where the bracket $\langle \ldots \rangle$ means the thermal average [Fig. 2(a)]. A convenient quantity to locate the critical point is Binder's cumulant defined by

$$U \equiv 1 - \frac{\langle |m|^4 \rangle}{3 \langle |m|^2 \rangle^2},$$

whose crossing point suggests $T_c \approx 2.45(2)$ in units of $J/k_B$ [Fig. 2(b)]. We extrapolate the magnetic order parameter under the assumption that [9]

$$\langle |m| \rangle \sim aN^{-\phi} + b \tag{2}$$

where the parameters $a, b$, and $\phi$ are found by the least-square fitting at each $T$. That is, we choose the fitting parameter $\phi$ that best describes $\langle |m| \rangle$ as a linear function of $N^{-\phi}$. We then observe that the limiting value of $\langle |m| \rangle$ at $N \to \infty$ could vanish at $T \gtrsim 2.4$ [Fig. 2(a)], which supports the above estimation of $T_c$. The exponent $\phi$ is found to have $\phi \approx 0.2$ around $T_c$.

In order to study the problem analytically, let us consider the block-spin transformation [1]. In Fig. 3(a), we illustrate the transformation in terms of $K$, which is indexed by the iteration step $n$. The bare coupling $K$ will be hence identified with $K_0$. This block-spin transformation replaces the upper spin block having two triangles and intermediate bonds by a single bond, mapping $K_n$ to $K_{n+1}$. We consider such a transformation at the outmost boundary layer so that an EBT with $L$ layers can be mapped to another EBT with $L - 1$

layers. The block-spin transformation can be performed by the majority rule at each triangle without ambiguity. The results are

$$\begin{aligned} e^{g+K_{n+1}} &= e^{5K+3K_n} + 2e^{-K+3K_n} + e^{-3K+3K_n} + 2e^{3K+K_n} \\ &\quad +2e^{-3K+K_n} + 2e^{3K-K_n} + 2e^{K-K_n} + 2e^{-3K-K_n} + 2e^{K-3K_n} \qquad (3) \\ &\equiv G_1(K, K_n), \\ e^{g-K_{n+1}} &= e^{3K+K_n} + 4e^{K+K_n} + 2e^{-K+K_n} \\ &\quad +e^{-5K+K_n} + 2e^{K-K_n} + 4e^{-K-K_n} + 2e^{-K-3K_n} \qquad (4) \\ &\equiv G_2(K, K_n), \end{aligned}$$

with a certain analytic function $g$, which appears as a consequence of removing short length scales. Therefore, one obtains a recursion relation

$$K_{n+1} = \frac{1}{2} \ln \left[ \frac{G_1(K, K_n)}{G_2(K, K_n)} \right], \qquad (5)$$

from Eqs. (3) and (4). We are interested in a fixed point $K_n = K_{n+1} = K_\infty$. Note that this fixed point is always stable, if it exists, since the slope at the crossing is less than 1 [Fig. 3(b)]. So it cannot be a point of phase separation, and we should instead ask ourselves when the renormalized coupling strength $K_\infty$ becomes infinite. It makes sense since it is equivalent to asking when the renormalized occupation probability becomes one in the bond-percolation problem [13]. The limiting $K_\infty$ diverges to infinity when $K \to K_c \approx 0.465\ 810$ [Fig. 3(b)]. However, this estimate corresponds to $T_c < 2.2$, which is not supported by the MC data above. We have also tried working with a larger spin block, but it hardly improves the estimate. In spite of such defects, which will be commented on again later, this block-spin transformation may give a glimpse of how to perform an RG analysis on this structure.

In order to find a recursion scheme explaining the MC results better, let us consider using transfer matrices. The idea is to describe exactly a single layer first and then replace it by a 1D chain. It is straightforward to write the transfer matrix for the 1D Ising chain with coupling strength $K_{n+1}$ as

$$T = \begin{pmatrix} e^{K_{n+1}} & e^{-K_{n+1}} \\ e^{-K_{n+1}} & e^{K_{n+1}} \end{pmatrix}, \qquad (6)$$

with two eigenvalues $\lambda_1^{(n+1)} = e^{K_{n+1}} + e^{-K_{n+1}} = 2\cosh K_{n+1}$ and $\lambda_2^{(n+1)} = e^{K_{n+1}} - e^{-K_{n+1}} = 2\sinh K_{n+1}$ [1]. On the other hand, the upper spin block in Fig. 4(a) has three spins and

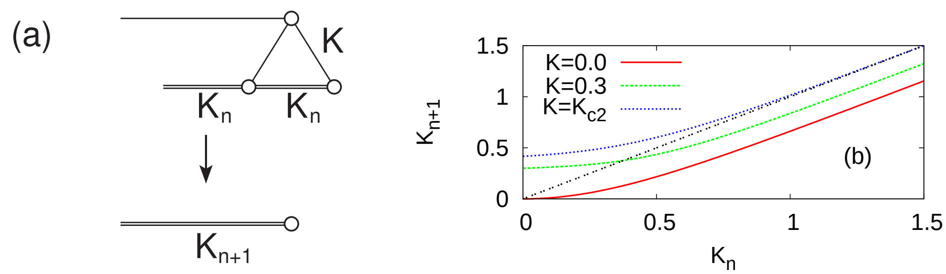


FIG. 4: (Color online) (a) Recursion scheme based on the transfer-matrix method. The double lines represent coarse-grained effective couplings. (b) Iteration by Eq. (7) at various $K$'s. The straight line has slope 1, i.e., $K_{n+1} = K_n$.

therefore yields an $8 \times 8$ transfer matrix $T'$ in terms of $K$ and $K_n$. The largest eigenvalue is obtained as

$$\lambda_1^{(n)} = \frac{1}{2} e^{-3K-4K_n} \left[ 2e^{4K+2K_n} + (e^{2K}+1)^2 e^{4K_n} + (e^{4K}+1) e^{2K+6K_n} + \sqrt{R_1(K,K_n)} \right],$$

with

$$\begin{aligned} R_1(K,K_n) &\equiv \left[ (e^{2K_n}+2)(e^{2K}+e^{2K_n}) e^{2K+2K_n} + e^{6K+6K_n} + e^{4K_n} \right]^2 \\ &\quad -4(e^{4K}-1)(e^{4K_n}-1)^2 e^{4K+4K_n}. \end{aligned}$$

The second largest eigenvalue is also available as

$$\lambda_2^{(n)} = \frac{1}{2} e^{-3K-4K_n} \left[ 2e^{4K+2K_n} - (e^{2K}+1)^2 e^{4K_n} + (e^{4K}+1) e^{2K+6K_n} + \sqrt{R_2(K,K_n)} \right],$$

with

$$\begin{aligned} R_2(K,K_n) &\equiv e^{6K+8K_n} \{ [-3\cosh K - \cosh 3K + 3\cosh(K-2K_n) + \cosh(3K+2K_n) \\ &\quad +4\cosh K_n \cosh(K+K_n) \sinh 2K]^2 - 32 \sinh 2K \sinh^2 2K_n \}. \end{aligned}$$

Since the matrix $T'$ has six more eigenvalues $\lambda_3^{(n)} > \lambda_4^{(n)} > \cdots > \lambda_8^{(n)}$, we may need to consider eight eigenvalues, or equivalently, eight energy levels from $E_1^{(n)} = -\ln \lambda_1^{(n)}$ to $E_8^{(n)} = -\ln \lambda_8^{(n)}$ in total. However, assuming that the first excitation from the ground state determines the most dominant behavior, we approximate this with a two-level system described by Eq. (6). According to this approximation, the first energy gap $E_2^{(n)} - E_1^{(n)}$ between the two lowest levels should be set equal to $E_2^{(n+1)} - E_1^{(n+1)} = -\ln \left[ \lambda_2^{(n+1)} / \lambda_1^{(n+1)} \right]$

in the simplified two-level picture. In short, we keep the ratio between the two largest eigenvalues at each iteration by

$$\frac{\lambda_2^{(n)}}{\lambda_1^{(n)}} = \frac{\lambda_2^{(n+1)}}{\lambda_1^{(n+1)}}, \tag{7}$$

which can be compared to Eq. (5) in our first attempt (see Ref. [15] where a similar idea is applied to the quantum Ising chain). It can be also interpreted as adjusting correlation lengths since the correlation length $\xi$ from a transfer-matrix calculation is given as

$$\xi^{(n)} = \frac{-1}{\ln\left[\lambda_2^{(n)}/\lambda_1^{(n)}\right]}. \tag{8}$$

Equation (7) defines a recursion relation for getting $K_{n+1}$ out of $K$ and $K_n$. We depict some cases of different $K$ values in Fig. 4(b). The limiting coupling strength $K_\infty$ becomes infinite when $K \to K_c \approx 0.416\ 550\ 7$, which corresponds to $T_c \approx 2.400\ 668$. Therefore, compared to the previous block-spin transformation, this approach describes the MC data better.

From Eq. (8), we can see how the correlation length $\xi$ along the boundary behaves as $K$ approaches $K_c$. The behavior turns out to be

$$\xi = \frac{-1}{\ln\left[\tanh K_\infty(K)\right]} \sim |K - K_c|^{-1/2}. \tag{9}$$

It is related to correlation observed in a sufficiently inner part of the system, but mediated by the boundary. The correlation-length exponent $\nu = 1/2$ in Eq. (9) indicates the mean-field result, which is consistent with the prediction in Ref. [8] and the MC analysis in Refs. [3, 16]. We again note that a number of boundary layers are mapped to a 1D chain that we are looking at, and that Eq. (9) is obtained in this respect. Although traveling along a single boundary layer is not the shortest path between a pair of spins, the *renormalized* boundary does contain the shortest path so that the mean-field exponent is found in this RG sense: it has been argued that the actual bare correlation will be an exponentially decaying function due to the radius of curvature [8]. By introducing magnetic field $h_1$ at this renormalized boundary part, we find

$$\lambda_1 = e^{K_\infty} \cosh \beta h_1 + \sqrt{e^{2K_\infty} \sinh^2 \beta h_1 + e^{-2K_\infty}},$$

which leads to the local susceptibility at $h_1 = 0$ as

$$\chi_1 = \frac{1}{\beta \lambda_1} \frac{\partial^2 \lambda_1}{\partial h_1^2}\bigg|_{h_1=0} = \beta e^{2K_\infty}.$$

From $K_\infty(K)$ shown in Eq. (9), it is straightforward to find

$$\chi_1 \sim |K - K_c|^{-1/2}. \tag{10}$$

In the surface critical phenomena [17], the correlation decays as $G_\parallel(r) \sim r^{-d+2-\eta_\parallel}$ in the parallel direction to the free surface, while it decays as $G_\perp(r) \sim r^{-d+2-\eta_\perp}$ in the perpendicular direction. According to the RG theory, two-spin bulk correlation $G(r) \sim r^{-d+2-\eta}$ is related to a scaling dimension $x_h$ by $G(r) \sim r^{-2x_h}$, where $x_h = d - y_h$ with the RG eigenvalue $y_h$. Suppose that the boundary scaling operator has another scaling dimension $x_{h,s}$. Then $G_\parallel(r) \sim r^{-2x_{h,s}}$ and $G_\perp(r) \sim r^{-x_h - x_{h,s}}$ depending on where the spins lie, and it is therefore predicted that $\eta_\perp = (\eta + \eta_\parallel)/2$. The local susceptibility diverges as $|K - K_c|^{-\gamma_1}$ with $\gamma_1 = \nu(2 - \eta_\perp)$. At the upper critical dimension, $\eta_\parallel = 2$ and $\eta = 0$, and the exponent $\gamma_1$ is thus predicted to be 1/2 in the mean-field theory. Therefore, Eq. (10) is fully consistent with the mean-field description as well as Eq. (9). This mean-field result also gives a clue to the poor performance of the block-spin transformation above: the magnetization distribution gets more and more spread instead of having sharp double peaks around $\pm 1$ as the dimension increases, so the concept of a block spin becomes less adequate.

In summary, we have studied the Ising model on top of the EBT structure by using the RG methods combined with the transfer-matrix calculation of its single layer. It is an approximate calculation considering only the two largest eigenmodes at each iteration, and it is hard to assess the error in this approximation, as is usual in many real-space RG methods. We have nevertheless reasonably predicted the order-disorder transition point, which suggests that this approximate calculation can capture some essential features of the system. Our main finding is that the critical behavior predicted from this analysis is consistent with the mean-field theory of free surfaces. Since many authors have expected that the bulk part far away from the boundary will exhibit the bulk mean-field transition, our RG analysis supports the mean-field picture all the way up to the boundary part in understanding the Ising model on a hyperbolic plane.

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[1] M. Plischke and B. Bergersen, *Equilibrium Statistical Physics*, 2nd ed. (World Scientific, Singapore, 1994)

[2] F. Sausset and G. Tarjus, J. Phys. A **40**, 12873 (2007).

[3] H. Shima and Y. Sakaniwa, J. Phys. A **39**, 4921 (2006).

[4] K. Ueda, R. Kcrmar, A. Gendiar, and T. Nishino, J. Phys. Soc. Jpn. **76**, 084004 (2007).

[5] S. K. Baek, P. Minnhagen, H. Shima, and B. J. Kim, Phys. Rev. E **80**, 011133 (2009).

[6] R. Lyons, J. Math. Phys. **41**, 1099 (2000).

[7] R. Méllin, J. C. A. d'Auriac, P. Chandra, and B. Douçot, J. Phys. A **29**, 5773 (1996).

[8] J. C. A. d'Auriac, R. Méllin, P. Chandra, and B. Douçot, J. Phys. A **34**, 675 (2001).

[9] S. K. Baek, P. Minnhagen, and B. J. Kim, Phys. Rev. E **79**, 011124 (2009).

[10] H. Shima and Y. Sakaniwa, J. Stat. Mech.: Theory Exp. P08017 (2006).

[11] T. Nogawa and T. Hasegawa, J. Phys. A **42**, 145001 (2009).

[12] P. Minnhagen and S. K. Baek, Phys. Rev. E **82**, 011113 (2010).

[13] S. K. Baek and P. Minnhagen, Physica A **390**, 1447 (2011).

[14] U. Wolff, Phys. Rev. Lett. **62**, 361 (1989).

[15] S. D. Drell, M. Weinstein, and S. Yankielowicz, Phys. Rev. D **16**, 1769 (1977).

[16] Y. Sakaniwa and H. Shima, Phys. Rev. E **80**, 021103 (2009).

[17] M. Henkel, *Conformal Invariance and Critical Phenomena* (Springer-Verlag, Berlin, 2010).